

DEHN TWISTS HAVE ROOTS

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Let S_g denote a closed, connected, orientable surface of genus g , and let $\text{Mod}(S_g)$ denote its mapping class group, that is, the group of homotopy classes of orientation preserving homeomorphisms of S_g .

Fact. If $g \geq 2$, then every Dehn twist in $\text{Mod}(S_g)$ has a nontrivial root.

It follows from the classification of elements in $\text{Mod}(S_1) \cong \text{SL}(2, \mathbb{Z})$ that Dehn twists are primitive in the mapping class group of the torus.

For Dehn twists about separating curves, the fact is well-known: if c is a separating curve then a square root of the Dehn twist T_c is obtained by rotating the subsurface of S_g on one side of c through an angle of π . In the case of nonseparating curves, the issue is more subtle. We give two (equivalent) constructions of roots below.

Geometric construction. Fix $g \geq 2$. Let P be a regular $(4g - 2)$ -gon. Glue opposite sides to obtain a surface $T \cong S_{g-1}$. The rotation of P about its center through angle $2\pi g/(2g - 1)$ induces a periodic map f of T . Notice that f fixes the points $x, y \in T$ that are the images of the vertices of P . Let T' be the surface obtained from T by removing small open disks centered at x and y . Define $f' = f|T'$.

Let A and B be annular neighborhoods of the boundary components of T' . Modify f' by an isotopy supported in $A \cup B$ so that

- $f'|_{\partial T'}$ is the identity,
- $f'|_A$ is a $g/(2g - 1)$ -left Dehn twist, and
- $f'|_B$ is a $(g - 1)/(2g - 1)$ -right Dehn twist.

Identify the two components of $\partial T'$ to obtain a surface $S \cong S_g$ and let $h : S \rightarrow S$ be the induced map. Then h^{2g-1} is a left Dehn twist along the gluing curve, which is nonseparating.

Algebraic construction. Let c_1, \dots, c_k be curves in S_g where c_i intersects c_{i+1} once for each i , and all other pairs of curves are disjoint. If k is odd, then a regular neighborhood of $\cup c_i$ has two boundary components, say, d_1 and d_2 , and we have a relation in $\text{Mod}(S_g)$ as follows:

$$(T_{c_1}^2 T_{c_2} \cdots T_{c_k})^k = T_{d_1} T_{d_2}.$$

This relation comes from the Artin group of type B_n , in particular, the factorization of the central element in terms of standard generators [2]. In the case $k = 2g - 1$, the curves d_1 and d_2 are isotopic nonseparating curves; call this isotopy class d . Using the fact that T_d commutes with each T_{c_i} , we see that

$$[(T_{c_1}^2 T_{c_2} \cdots T_{c_{2g-1}})^{1-g} T_d]^{2g-1} = T_d.$$

Date: October 28, 2008.

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Other roots. All roots of Dehn twists are obtained in a similar way. That is, if f is a root of a Dehn twist T_d then the canonical reduction system for f is d [1]. By the Nielsen–Thurston classification for surface homeomorphisms [3], if we cut the surface along d , then f restricts to a finite order element.

Roots of half-twists. Let $S_{0,2g+2}$ be the sphere with $2g + 2$ punctures (or cone points) and let d be a curve in $S_{0,2g+2}$ with 2 punctures on one side and $2g$ on the other. On the side of d with 2 punctures, we perform a left half-twist, and on the other side we perform a $(g - 1)/(2g - 1)$ -right Dehn twist by arranging the punctures so that one puncture is in the middle, and the other punctures rotate around this central puncture. The $(2g - 1)^{\text{st}}$ power of the composition is a left half-twist about d . Thus, we have roots of half-twists in $\text{Mod}(S_{0,2g+2})$ for $g \geq 2$. There is a 2-fold orbifold covering $S_g \rightarrow S_{0,2g+2}$ where the relation from our algebraic construction above descends to this relation in $\text{Mod}(S_{0,2g+2})$. A slight generalization of this construction gives roots of half-twists in any $\text{Mod}(S_{0,n})$ with $n \geq 5$.

Roots of elementary matrices. If we consider the map $\text{Mod}(S_g) \rightarrow \text{Sp}(2g, \mathbb{Z})$ given by the action of $\text{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z})$, we also see that elementary matrices in $\text{Sp}(2g, \mathbb{Z})$ have roots; for instance, we have

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By stabilizing, we obtain cube roots of elementary matrices in $\text{Sp}(2g, \mathbb{Z})$ for $g \geq 2$.

Roots of Nielsen transformations. Let F_n denote the free group generated by x_1, \dots, x_n , let $\text{Aut}(F_n)$ denote the group of automorphisms of F_n , and assume $n \geq 2$. A Nielsen transformation in $\text{Aut}(F_n)$ is an element conjugate to the one given by $x_1 \mapsto x_1 x_2$ and $x_k \mapsto x_k$ for $2 \leq k \leq n$. The following automorphism is the square root of a Nielsen transformation in $\text{Aut}(F_n)$ for $n \geq 3$.

$$\begin{aligned} x_1 &\mapsto x_1 x_3 \\ x_2 &\mapsto x_3^{-1} x_2 x_3 \\ x_3 &\mapsto x_3^{-1} x_2 \end{aligned}$$

Taking quotients, this gives a square root of a Nielsen transformation in $\text{Out}(F_n)$ and, multiplying by $-\text{Id}$, a square root of an elementary matrix in $\text{SL}(n, \mathbb{Z})$, $n \geq 3$. Finally, our roots of Dehn twists in $\text{Mod}(S)$ can be modified to work for punctured surfaces, thus giving “geometric” roots of Nielsen transformations in $\text{Out}(F_n)$.

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